

On perfect k -matchings *

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Abstract

In this paper, we generalize the notions of perfect matchings, perfect 2-matchings to perfect k -matchings and give a necessary and sufficient condition for existence of perfect k -matchings. For bipartite graphs, we show that this k -matching problem is equivalent to that matching question. Moreover, for regular graphs, we provide a sufficient condition of perfect k -matching in terms of edge connectivity.

Keywords: matching; 2-matching; k -matching.

1 Introduction

All graphs considered are multigraphs (with loops) and finite. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices of a graph G is called the *order* of G . Unless otherwise defined, we follow [3] for terminologies and definitions.

We denote the degree of vertex v in G by $d_G(v)$. For two subsets $S, T \subseteq V(G)$, let $e_G(S, T)$ denote the number of edges of G joining S to T . For a set X , we denote the cardinality of X by $|X|$. A vertex of degree zero is called an *isolated vertex*. Let $Iso(G)$ denote the set of isolated vertices of G and let $i(G) = |Iso(G)|$. Let $c_o(G)$ denote the number of odd components of G . Let $odd(G)$ denote the number of odd components with order at least three of G . For any subset X of vertices of G , we define the neighbourhood of X in G to be the set of all vertices adjacent to vertices in X ; this set is denoted by $N_G(X)$.

A *matching* M of a graph G is a subset of $E(G)$ such that any two edges of M have no end-vertices in common. Let k be a positive. A k -*factor* of a graph G is a spanning subgraph H of G such that $d_H(x) = k$ for every $x \in V(G)$. A $\{K_2, C_{2t+1} \mid t \geq 1\}$ -*factor* of

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a graph G is a spanning subgraph of G such that each of its components is isomorphic to one of $\{K_2, C_{2t+1} \mid t \geq 1\}$.

Let $f : \{0, 1, \dots, k\} \rightarrow E(G)$ be an assignment such that the sum of weights of edges incident with any vertex is at most k , i.e., $\sum_{e \sim v} f(e) \leq k$ for any vertex $v \in V(G)$. A k -matching is a subgraph induced by the edges with weight among $1, \dots, k$ such that $\sum_{e \sim v} f(e) \leq k$. The sum of all weights, i.e., $\sum_{e \in E(G)} f(e)$, is called *size* of a k -matching f . A k -matching is *perfect* if $\sum_{e \sim v} f(e) = k$ for every vertex $v \in V(G)$. Clearly, a k -matching is perfect if and only if its size is $k|V(G)|/2$. If $k = 1$, then a perfect k -matching is called a *perfect matching*. If $k = 2$, then a perfect k -matching is called a *perfect 2-matching*.

For perfect matching of bipartite graphs, Hall obtained the next result in terms of isolated vertices.

Theorem 1.1 (Hall, [2]) *Let $G = (X, Y)$ be a bipartite graph. Then G has a perfect matching if and only if $|X| = |Y|$ and for any $S \subseteq X$,*

$$i(G - S) \leq |S|.$$

Tutte (1947) studied the perfect matching of general graphs and gave the sufficient and necessary condition.

Theorem 1.2 (Tutte, [4]) *A graph G has a perfect matching if and only if for any $S \subseteq V(G)$,*

$$c_o(G - S) \leq |S|.$$

For perfect 2-matching, Tutte (1953) gave the following result.

Theorem 1.3 (Tutte, [6]) *Let G be a connected graph. Then the following statements are equivalent:*

- (1) G has a perfect 2-matching;
- (2) $i(G - S) \leq |S|$ for all subsets $S \subseteq V(G)$;
- (3) G has a $\{K_2, C_{2t+1} \mid t \geq 1\}$ -factor.

In the proof, we need the following technical theorems.

Theorem 1.4 (Tutte, [5]) *Let G be a graph and k a positive integer. Then G has a k -factor if and only if, for all $D, S \subseteq V(G)$ with $D \cap S = \emptyset$,*

$$\delta_G(D, S) = k|D| - k|S| + \sum_{v \in S} d_{G-D}(v) - \tau_G(S, T) \geq 0,$$

where $\tau_G(D, S)$ is the number of components C of $G - (D \cup S)$ such that $e_G(V(C), S) + k|C| \equiv 1 \pmod{2}$. Moreover, $\delta_G(D, S) \equiv k|V(G)| \pmod{2}$.

2 Main Results

In this section, we gave a good characterization for perfect k -matchings.

Theorem 2.1 *Let $k \geq 4$ be even. Then G contains a perfect k -matching if and only if G contains a perfect 2-matching.*

Proof. Suppose that G contains a perfect 2-matching. By Theorem 1.3, G contains a $\{K_2, C_{2l+1}\}$ -factor H . We assign every isolated edge of H with weight k and the rest edge with weight $k/2$. Then we obtain a perfect k -matching of G .

Conversely, suppose G that contains a perfect k -matching H . Then there exists a function $f : V(G) \rightarrow \{0, 1, \dots, k\}$ such that $\sum_{v \sim e} f(e) = k$ for all $v \in V(G)$. We claim $i(G - S) \leq |S|$ for all $S \subseteq V(G)$. Otherwise, assume that there exists $S \subseteq V(G)$ such that $i(G - S) > |S|$. Then we have

$$ki(G - S) = \sum_{e \in E_G(Iso(G-S), S)} f(e) > k|S|,$$

a contradiction. So by Theorem 1.3, G contains a perfect 2-matching. \square

Corollary 2.2 *Let $k \geq 2$ be even. Then a graph G contains a perfect k -matching if and only if $i(G - S) \leq |S|$ for all $S \subseteq V(G)$.*

Theorem 2.3 *Let $k \geq 1$ be odd. Then G contains a perfect k -matching if and only if*

$$odd(G - S) + ki(G - S) \leq k|S| \quad \text{for all subsets } S \subseteq V(G).$$

Proof. We first prove the necessity. Suppose that G has a perfect k -matching and there exists $S \subseteq V(G)$ such that

$$odd(G - S) + ki(G - S) > k|S|.$$

Let $f : E(G) \rightarrow \{0, 1, \dots, k\}$ such that $\sum_{e \sim v} f(e) = k$ for all $v \in V(G)$. Let $m = odd(G - S)$ and let C_1, \dots, C_m denote the odd components of $G - S$ with order at least three. Let $W = C_1 \cup \dots \cup C_m$. Since k is odd, by parity, every odd component with order at least

three can't contain a perfect k -matching. So $\sum_{e \in E_G(V(C_i), S)} f(e) \geq 1$ for $i = 1, \dots, m$. Then we have

$$\begin{aligned} k|S| &= \sum_{v \in S} \sum_{e \sim v} f(e) \geq \sum_{e \in E_G(V(W), S)} f(e) + \sum_{e \in E_G(Iso(G-S), S)} f(e) \\ &\geq \text{odd}(G-S) + ki(G-S) > k|S|, \end{aligned}$$

a contradiction. So the result is followed.

We next prove the sufficiency. Let G^* be obtained from G by changing every edge of G into k parallel edges. Then G contains a perfect k -matching if and only if G^* contains a k -factor. Conversely, suppose that G contains no perfect k -matchings. Then G^* contains no k -factors. By Theorem 1.4, there exist two disjoint subset $D, S \subseteq V(G^*)$ such that

$$k|D| - k|S| + \sum_{x \in S} d_{G^*-D}(x) - \tau < 0,$$

where τ denote the number of components C of G^*-D-S such that $k|V(C)| + e_{G^*}(V(C), S) \equiv 1 \pmod{2}$. Let C_1, \dots, C_τ denote those components and $W = \bigcup_{i=1}^\tau C_i$. By Theorem 1.2, we can suppose that $k \geq 3$.

Without loss of generality, among all such subsets, we choose subsets D and S such that S is minimal. We have $S \neq \emptyset$, otherwise, $k|D| < \tau$ and $|V(C_i)|$ is odd for $i = 1, \dots, \tau$. So we have

$$k|D| - ki(G-D) - \text{odd}(G-D) \leq k|D| - \tau < 0,$$

a contradiction. Let $M = G^* - D - S - V(W)$.

Claim 1. $G[S]$ consists of isolated vertices.

Otherwise, let $e = uv \in G[S]$. Let $|N_G(v) \cap V(W)| = m$. Let $D' = D$ and $S' = S - v$. Let τ' denote the number of components C of $G^*-D'-S'$ such that $k|V(C)| + e_{G^*}(V(C), S') \equiv 1 \pmod{2}$. Then we have

$$\begin{aligned} k|D'| - k|S'| + \sum_{x \in S'} d_{G^*-D'}(x) - \tau' &\leq k|D| - k(|S| - 1) + \sum_{x \in S-v} d_{G^*-D}(x) - (\tau - m) \\ &\leq k|D| - k|S| + k + \sum_{x \in S} d_{G^*-D}(x) - d_{G^*-D}(v) - (\tau - m) \\ &\leq k|D| - k|S| + k + \sum_{x \in S} d_{G^*-D}(x) - k(m+1) - \tau + m \\ &\leq k|D| - k|S| + \sum_{x \in S} d_{G^*-D}(x) - \tau < 0, \end{aligned}$$

contradicting to the minimality of S . This completes the claim.

With the similar proof of Claim 1, we obtain the following claim.

Claim 2. $e_G(S, V(M)) = \emptyset$.

Claim 3. $|N_G(x) \cap V(W)| \leq 1$ for all $x \in S$.

Otherwise, suppose that there exists $v \in S$ such that $m = |N_G(v) \cap V(W)| \geq 2$. Let $D'' = D$ and $S'' = S - v$. Let τ'' denote the number of components C of $G^* - D'' - S''$ such that $k|V(C)| + e_{G^*}(V(C), S'') \equiv 1 \pmod{2}$. Then we have

$$\begin{aligned}
k|D''| - k|S''| + \sum_{x \in S''} d_{G^* - D''}(x) - \tau'' &\leq k|D| - k(|S| - 1) + \sum_{x \in S - v} d_{G^* - D}(x) - (\tau - m) \\
&\leq k|D| - k|S| + k + \sum_{x \in S} d_{G^* - D}(x) - d_{G^* - D}(v) - (\tau - m) \\
&\leq k|D| - k|S| + k + \sum_{x \in S} d_{G^* - D}(x) - km - \tau + m \\
&= k|D| - k|S| + \sum_{x \in S} d_{G^* - D}(x) - \tau - (k - 1)(m - 1) + 1 \\
&\leq k|D| - k|S| + \sum_{x \in S} d_{G^* - D}(x) - \tau < 0,
\end{aligned}$$

contradicting to the minimality of S . This completes the claim.

Claim 4. $E_G(S, V(W)) = \emptyset$.

Otherwise, by Claim 2, suppose that there exists an edge $uv \in E_G(S, V(W))$, where $v \in S$ and $u \in V(W)$. Let $D''' = D$ and $S''' = S - v$. Let τ''' denote the number of components C of $G^* - D''' - S'''$ such that $k|V(C)| + e_{G^*}(S''', V(C)) \equiv 1 \pmod{2}$. Without loss of generality, suppose that $u \in C_1$. By Claims 1, 2 and 3, then $G^*[V(C_1) \cup \{v\}]$ is a component of $G^* - D''' - S'''$. Note that $k|V(C_1) \cup \{v\}| + e_{G^*}(S''', V(C_1) \cup \{v\}) \equiv k|V(C_1)| + e_{G^*}(S, V(C_1)) \equiv 1 \pmod{2}$. So $\tau = \tau'''$. Hence

$$\begin{aligned}
k|D'''| - k|S'''| + \sum_{x \in S'''} d_{G^* - D'''}(x) - \tau''' &= k|D| - k(|S| - 1) + \sum_{x \in S - v} d_{G^* - D}(x) - \tau \\
&\leq k|D| - k|S| + k + \sum_{x \in S} d_{G^* - D}(x) - d_{G^* - D}(v) - \tau \\
&\leq k|D| - k|S| + \sum_{x \in S} d_{G^* - D}(x) - \tau < 0,
\end{aligned}$$

contradicting to the minimality of S . This completes the claim.

Since $e_{G^*}(V(C_i), S) + k|V(C_i)| \equiv 1 \pmod{2}$ and k is odd, by Claim 4, we have $|V(C_i)| \equiv 1 \pmod{2}$ for $i = 1, \dots, \tau$. By Claims 1, 2, and 4, we have

$$\begin{aligned} 0 &> k|D| - k|S| + \sum_{x \in S} d_{G^*-D}(x) - \tau \\ &= k|D| - k|S| - \tau \\ &\geq k|D| - ki(G - D) - \text{odd}(G - D). \end{aligned}$$

Hence we have $k|D| < ki(G - D) + \text{odd}(G - D)$, a contradiction. We complete the proof. \square

Theorem 2.4 *Let $G = (U, W)$ be a bipartite graph, where $|U| = |W|$. Then G contains a perfect matching if and only if G contains a perfect k -matching.*

Proof. Necessity is obvious. Now we prove the sufficiency. Suppose that G contains a perfect k -matching. Let $f : E(G) \rightarrow \{0, 1, \dots, k\}$ such that $\sum_{v \sim e} f(e) = k$ for all $v \in V(G)$. Then for all independent set S , we have

$$k|S| = \sum_{v \in S} \sum_{v \sim e} f(e) \leq \sum_{e \in E_G(S, N(S))} f(e) \leq k|N(S)|. \quad (1)$$

So we have $i(G - S) \leq |S|$ for all $S \subseteq U$. By Theorem 1.1, G contains a perfect matching. This completes the proof. \square

Corollary 2.5 *Let $k \geq 1$ be an odd integer and G be an r -regular, λ -edge-connected graph. Suppose that*

$$\lambda = \begin{cases} \lceil \frac{r}{k} \rceil - 1 & \text{if } \lceil \frac{r}{k} \rceil \equiv r \pmod{2}; \\ \lceil \frac{r}{k} \rceil & \text{if } \lceil \frac{r}{k} \rceil \not\equiv r \pmod{2}. \end{cases}$$

Then G contains a perfect k -matching.

Proof. Suppose that the result doesn't hold. By Theorem 2.3, there exists a subset $S \subseteq V(G)$ such that

$$\text{odd}(G - S) + ki(G - S) > k|S|.$$

Let $m = \text{odd}(G - S)$. Let C_1, \dots, C_m denote these odd components with order at least three of $G - S$. Since $r|C_i| - e_G(V(C_i), S) = \sum_{x \in V(C_i)} d_{C_i}(x)$ is even, so $e_G(V(C_i), S) \equiv r$. Since G is an r -regular, λ -edge-connected graph, so if $\lceil \frac{r}{k} \rceil \equiv r \pmod{2}$, then $e_G(V(C_i), S) \geq \lambda + 1$ for $i = 1, \dots, m$. So we have

$$r|S| \geq \lceil \frac{r}{k} \rceil \text{odd}(G - S) + ri(G - S).$$

Hence,

$$\begin{aligned} kr|S| &\geq k\lceil \frac{r}{k} \rceil \text{odd}(G - S) + kri(G - S) \\ &\geq r\text{odd}(G - S) + kri(G - S) > kr|S|, \end{aligned}$$

a contradiction. This completes the proof. \square

Corollary 2.6 (Bäbler, [1]) *Let G be an r -regular, $(r-1)$ -edge-connected graph with even order. Then G contains a perfect matching.*

References

- [1] F. Bäbler, Über die zerlegung regulärer streckenkomplexe ungerader ordnung, *Comment. Math. Helvetici*, **10** (1938), 275-287.
- [2] P. Hall, On representatives of subsets, *J. London Math. Soc.*, **10** (1935), 26-30.
- [3] L. Lovász and M. D. Plummer, Matching Theory, *Ann. Discrete Math.*, **29** North-Holland, Amsterdam, 1986.
- [4] W. T. Tutte, The factorization of linear graphs, *J. London Math. Soc.*, **22** (1947), 107-111.
- [5] W. T. Tutte, The factors of graphs, *Can. J. Math.*, **4** (1952), 314-328.
- [6] W. T. Tutte, The 1-factors of oriented graphs, *Proc. Amer. Math. Soc.*, **4** (1953), 922-931.